Roll No. ....

E–992

M. A./M. Sc. (Fourth Semester) (Main/ATKT) EXAMINATION, May-June, 2021

MATHEMATICS

Paper First

(Functional Analysis—II)

Time : Three Hours] [*Maximum Marks : 80*

Note : Attempt all Sections as directed.

Section—A 1 each

(Objective/Multiple Choice Questions)

Note : Attempt all questions.

Choose the correct answer :

1. Let X be a Banach space and Y be a normed linear space over field K. If F is bounded linear operator from X into Y, then F is :

   (a) Bounded

   (b) Continuous

   (c) Uniformly bounded

   (d) Unbounded

2. Let X and Y be Banach space and $T \in B(X, Y)$. If T is onto, then there exist $k > 0$ such that for every $y \in Y$ there exist $x \in X$ such that :

   (a) $Tx \geq k\|y\|$

   (b) $Tx \leq k\|y\|$

   (c) $Tx = k\|y\|$

   (d) None of the above

3. Let N and N' be normed linear space and $D \subset N$. Then a linear transformation $T : D \to N'$ is closed if and only if G_T is :

   (a) Open

   (b) Closed

   (c) Both open and closed

   (d) None of the above

4. Let X be a complex normed linear space and let M a linear subspace of X. If $f \in M^*$, then there exist $g \in X^*$ such that :

   (a) $f \leq g$ and $\|g\| \neq \|f\|$

   (b) $f \leq g$ and $\|g\| = \|f\|$

   (c) $f \geq g$ and $\|g\| \neq \|f\|$

   (d) $f \geq g$ and $\|g\| = \|f\|$

5. A normed linear space X is reflexive, if :
   (a) J (X) = X**
   (b) J (X) = X
   (c) J (X) = X*
   (d) J (X) $\neq$ X**

6. If S is a subspace of normed linear space X, then :
   (a) $S = S^{\perp}$
   (b) $\overline{S}' = S^{\perp}$
   (c) $\overline{S} = S^{\perp}_{\perp}$
   (d) $S' = S^{\perp}_{\perp}$

7. If X is a Banach space and $S_r\left(x\right)$ denotes open sphere of radius centered at $x \in X$, then :
   (a) $S_r\left(x_0\right) = r.S_0$
   (b) $S_r\left(x_0\right) = x_0 - S_r$
   (c) $S_r\left(x_0\right) = x_0.S_r$
   (d) $S_r\left(x_0\right) = x_0 + S_r$

8. Let T be a closed linear map of a Banach space X into a Banach space Y. Then T is :
   (a) Isomorphic
   (b) Continuous
   (c) Uniformly continuous
   (d) Homomorphism

9. If x and y are any *two* vectors in an inner product space X, then :

$$\|x + y\| \leq \|x\| + \|y\|$$

   (a) Cauchy-Schwarz's inequality
   (b) Parallelogram law
   (c) Triangle inequality
   (d) Bessel's inequality

10. The sign of equality holds in the inequality $\left|\langle x, y\rangle\right| = \|x\|\|y\|$ if and only if :
   (a) x and y are linearly independent.
   (b) x and y are linearly dependend.
   (c) x is linearly dependent and y is linearly independent.
   (d) x is linearly independent and y is linearly dependent.

11. Let X be an inner product space. Two elements x and y are said to be orthogonal, if :
   (a) $\langle x, y\rangle = 0$
   (b) $\langle x, y\rangle \neq 0$
   (c) $\langle x, y\rangle > 0$
   (d) $\langle x, y\rangle < 0$

12. Let M be a linear subspace of Hilbert space X. Then M is closed if and only if :
   (a) $M \neq M^{\perp\perp}$
   (b) $M = M^{\perp}$
   (c) $M \neq M^{\perp}$
   (d) $M = M^{\perp\perp}$

13. Every Hilbert space is :
    (a) Reflexive
    (b) Symmetric
    (c) Transitive
    (d) All of the above

14. If T_1 and T_2 are normal operators on a Hilbert space H with the property that either it commutes with the adjoint of other, then $T_1 + T_2$ and $T_1 T_2$ are :
    (a) Self-adjoint
    (b) Normal
    (c) Unitary
    (d) Positive

15. If M is subspace of the Hilbert space H, then for any $x \in \mathrm{H}$ there exist a unique $y \in \mathrm{M}$ and $z \perp \mathrm{M}$ such that $x = y + z$. Then vector y is called :
    (a) Projection of x onto M
    (b) Linear of x onto M
    (c) Bounded linear of x onto M
    (d) None of the above

16. Let y be a fixed vector in a Hilbert space H and let f_y be a scalar valued function on H defined by $f_y(x) = \langle x, y \rangle \, \forall \, x \in \mathrm{H}$. Then f_y is functional in H^* and :
    (a) $\|y\| \le \|f_y\|$
    (b) $\|y\| \ge \|f_y\|$
    (c) $\|y\| = \|f_y\|$
    (d) $\|y\| \ne \|f_y\|$

17. If $\mathrm{T} \in \mathrm{B}(\mathrm{H})$, then there exist a unique $\mathrm{U} \in \mathrm{B}(\mathrm{H})$ such that :
    (a) $(\mathrm{T}x, y) = (x, \mathrm{U}_y) \, \forall \, x, y \in \mathrm{H}$

    (b) $(\mathrm{T}x, y) = (\mathrm{U}_x, y) \, \forall \, x, y \in \mathrm{H}$
    (c) $(x, \mathrm{T}y) = (x, \mathrm{U}_y) \, \forall \, x, y \in \mathrm{H}$
    (d) $(\mathrm{T}x, y) = (\mathrm{U}_x, y) \, \forall \, x, y \in \mathrm{H}$

18. An operator T on a Hilbert space H is unitary if and only if it is .... of H onto itself.
    (a) Isomorphism
    (b) Homeomorphism
    (c) Isometrically isomorphism
    (d) None of the above

19. **Statement I :** Every self-adjoint operator is normal but the converse is not true.

    Statement II : Every unitary operator is normal but the converse is not true.
    (a) Statement I is true and Statement II is false.
    (b) Statement I is false and Statement II is true.
    (c) Both Statements I and II are true.
    (d) Both Statements I and II are false.

20. $\mathrm{T} \in \mathrm{B}(\mathrm{H})$ is said to be .... if and only if real and imaginary parts commute.
    (a) Hermitian
    (b) Normal
    (c) Unitary
    (d) Positive

Section—B 2 each

(Very Short Answer Type Questions)

Note : Attempt all questions. Answer in 2 to 3 sentences.

1. State open mapping theorem.
2. Define uniformly convex of a normed linear space.
3. Define Parallelogram law.

4. Define Hilbert space with an example.
5. Let H be a Hilbert space and let $x, y \in \mathrm{H}$. If x is orthogonal to y, then prove that :

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2$$

6. Let T be an operator on H and T $\rightarrow$ T* is a mapping of B (H) into itself for $T_1, T_2 \in$ B (H). Prove that :

$$(T_1 + T_2)^* = T_1^* + T_2^*$$

7. Let T be a bounded linear operator on a Hilbert space H. Then prove that T is normal $\Leftrightarrow \|T^* x\| = \|Tx\| \forall x \in \mathrm{H}$.
8. State Generalized Lax-Milgram theorem.

Section—C 3 each

(Short Answer Type Questions)

Note : Attempt all questions. Answer in 75 words.

1. State and prove Triangle inequality.
2. In an inner product space, prove that the inner product space is jointly continuous.
3. State and prove Riesz-Fischer theorem.
4. Let H be a Hilbert space and let the mapping $\psi : \mathrm{H} \rightarrow \mathrm{H}^*$ be defined by $\psi(y) = f_y(x) = (x, y) \forall x \in \mathrm{H}$. Then prove that ψ is one-one, onto, additive but not linear and isometry.
5. State and prove Bessel's inequality.
6. Let S and T be unitary operators on a Hilbert space H. Then :
   (i) T is isometeric.
   (ii) T is normal.
   (iii) S_T is unitary.
7. Prove that an operator T on a Hilbert space H is unitary if and only if it is an isometric isomorphism of H onto itself.
8. Write the statement of Hahn-Banach theorem for real, complex and normed linear space.

Section—D 5 each

(Long Answer Type Questions)

Note : Attempt all questions. Answer in 150 words.

1. Let X and Y be Banach space and T be a continuous linear transformation of X onto Y. Then the image of each open sphere centered on the origin in X contain an open sphere centered on the origin in Y.

Or

Let X be a Banach space and Y be an normed linear space. Let $\{T_i\}$ be a non-empty set of continuous linear transformation form X into Y such that $\{T_i(x)\}$ is bounded for each x in X. Then $\{\|T_i\|\}$ is bounded.

2. Derive complex version of Hahn-Banach theorem.

Or

Let $\{T_n\}$ be a sequence of compact linear operator from a normed space X into a Banach space Y and T be a bounded linear operator T : X $\rightarrow$ Y such that $\|T_n - T\| \rightarrow 0$ as $n \rightarrow \infty$. Then the limit operator T is compact.

3. State and prove Riesz-Representation theorem.

Or

Prove that a Banach space is a Hilbert space if and only if its norm satisfies the parallelogram law.

4. If T is a positive operator on a Hilbert space H, then I + T is non-singular.

Or

Let X be a Banach space. Then X is reflexive if and only if X* is reflexive.